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# Random walks in a random environment on a strip: a renormalization group approach

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## Abstract

We present a real space renormalization group scheme for the problem of random walks in a random environment on a strip, which includes the one-dimensional random walk in a random environment with bounded non-nearest-neighbour jumps. We show that the model renormalizes to an effective one-dimensional random walk with nearest-neighbour jumps and conclude that Sinai scaling is valid in the recurrent case, while in the sub-linear transient phase, the displacement grows as a power of the time.

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## 1. Introduction

The problem of random walks in a random environment (RWRE) has a long history and since the early results in the 1970s [1], a vast amount of informations have accumulated; for a recent review see [2]. The RWRE can be regarded as a toy model of disordered systems, for which exact results are available and which, due to its simple formulation, became a fundamental model in various fields such as transport processes or statistical mechanics of magnetic systems [3]. Most works concern the RWRE with nearest-neighbour jumps on the integers, for which a more or less complete picture is at our disposal. Beside rigorous results [1, 4, 5], this model was also studied by a strong disorder renormalization group (SDRG) method [6] which is closely related to that originally developed for disordered spin models [7]. This method, in which the small barriers of the energy landscape are successively eliminated, yields exact results for the asymptotical dynamics, among others the scaling of the typical displacement  $x$  of the walker with time  $t$  in the recurrent case:  $x \sim (\ln t)^2$ , in accordance with Sinai's theorem [4].

In higher dimensions, even on quasi-one-dimensional lattices or in case of non-nearest-neighbour jumps, the understanding of RWRE is at present far from complete. For the one-dimensional (1D) RWRE with bounded non-nearest-neighbour jumps, criteria for recurrence

and transience are known [8] and for some special cases Sinai scaling was proven [9]. This model arises also in the context of disordered dynamical systems [10]. For the RWRE on strips of finite width, which incorporates among others the former model and the persistent RWRE [11], recurrence and transience criteria were obtained in [12].

The aim of this paper is to propose an exact SDRG scheme for the RWRE on a strip. A necessary condition for the analytical tractability by the SDRG method is that the topology of the underlying lattice is invariant under the transformation, which generally does not hold apart from 1D. As in our approach complete layers of lattice sites are decimated, the topology of the network of transitions is preserved. Contrary to the 1D RWRE, the energy landscape does not exist in general, therefore we keep track of the transformation of jump rates in the same spirit as it was done for the closely related 1D zero-range process [13]. We shall show that in the fixed point, the transformation of relevant variables is identical to that of the 1D RWRE with nearest-neighbour jumps, implying among others that Sinai scaling holds for strips of finite width in the recurrent case.

The rest of the paper is organised as follows. In section 2, the problem to be studied is defined in details. In section 3, the renormalization group (RG) transformation is introduced and the RG equations are analysed in the recurrent case, as well as in the zero-velocity transient phase. Finally, the results are discussed in section 4.

## 2. Formulation of the problem

We consider a finite strip  $S = \{1, \dots, L\} \times \{1, \dots, m\}$  of length  $L$  and width  $m$ , and call the set of sites  $(n, i) \in S$  with fixed  $n$  and  $i = 1, \dots, m$  the  $n$ th layer. We define on this lattice a continuous-time random walk by the following (non-negative) transition rates for  $1 \leq n \leq L$ :

$$T(z_1, z_2) = \begin{cases} P_n(i, j) & \text{if } z_1 = (n, i), z_2 = (n + 1, j) \\ Q_n(i, j) & \text{if } z_1 = (n, i), z_2 = (n - 1, j) \\ R_n(i, j) & \text{if } z_1 = (n, i), z_2 = (n, j), i \neq j \\ 0 & \text{otherwise.} \end{cases}$$

Here and in the following, the formally appearing index  $(0, j)$  [ $(L + 1, j)$ ] is meant to refer to site  $(L, j)$  [ $(1, j)$ ], i.e. the strip is periodic in the first coordinate. The  $m \times m$  matrix  $P_n$  ( $Q_n$ ) contains the jump rates from the  $n$ th layer to the adjacent layer on the right(left), while the matrix  $R_n$  with diagonal elements  $R_n(i, i) := -\sum_{j \neq i} R_n(i, j)$  contains the intra-layer jump rates. Besides, we define the  $m \times m$  matrix  $S_n$ , which will be useful in later calculations by  $S_n(i, j) := -R_n(i, j)$ ,  $i \neq j$ , while the diagonal elements are fixed by

$$(P_n + Q_n - S_n)\mathbf{1} = \mathbf{0}, \tag{1}$$

where  $\mathbf{1}(\mathbf{0})$  is a column vector with all components 1(0). For the sequence of triples of matrixes,  $\{(P_n, Q_n, R_n)\}$ , which defines the random environment, we impose at this point the only condition that it must be connected in the sense that every site is reachable from every other site through sequences of consecutive transitions with positive rates. The probability that the walker resides on site  $(n, i)$  in the stationary state is denoted by  $\pi_n(i)$  and these are normalized as  $\sum_{(n,i)} \pi_n(i) = 1$ . Following [12], we introduce the row vectors  $\pi_n = (\pi_n(i))_{1 \leq i \leq m}$  and for a fixed environment, write the system of linear equations that the stationary probabilities satisfy in the form

$$\pi_n S_n = \pi_{n-1} P_{n-1} + \pi_{n+1} Q_{n+1}, \quad 1 \leq n \leq L. \tag{2}$$

Although, we started from a continuous-time random walk, the same equations can be written for a discrete-time jump process with transition probabilities obtained by rescaling the transition rates by  $\max_{(n,i)} S_n(i, i)$ .

### 3. Renormalization group transformation

The elementary step of the renormalization group method we apply is the elimination of the  $k$ th layer, such that the walker then jumps from the  $k - 1$ st layer directly to the  $k + 1$ st one with transition rates  $\tilde{P}_{k-1}(i, j)$  and from the  $k + 1$ st layer to the  $k - 1$ st one with rates  $\tilde{Q}_{k+1}(i, j)$ . We choose the matrices  $\tilde{P}_{k-1}$  and  $\tilde{Q}_{k+1}$  in such a way that the remaining  $L - 1$  equations in (2) are fulfilled by the unchanged vectors  $\pi_n, n \neq k$ . Eliminating  $\pi_k$  in equation (2), it turns out that also the matrices  $S_{k-1}$  and  $S_{k+1}$  must be changed, and we have the following transformation rules:

$$\tilde{P}_{k-1} = P_{k-1} S_k^{-1} P_k \quad (3)$$

$$\tilde{Q}_{k+1} = Q_{k+1} S_k^{-1} Q_k \quad (4)$$

$$\tilde{S}_{k-1} = S_{k-1} - P_{k-1} S_k^{-1} Q_k \quad (5)$$

$$\tilde{S}_{k+1} = S_{k+1} - Q_{k+1} S_k^{-1} P_k. \quad (6)$$

All other matrices remain unchanged. The matrix  $S_n$  has the following important property:

$$S_n^{-1} \geq 0, \quad (7)$$

which is meant to hold for the matrix elements. This can be proven as follows. We introduce the notation  $D_m \equiv \det S_n$  where the index  $m$  refers to the order of the matrix. The non-diagonal elements of  $S_n$  are nonpositive, while  $S_n(i, i) \equiv \sum_j [P_n(i, j) + Q_n(i, j)] + \sum_{j \neq i} R_n(i, j) > 0$  for  $1 \leq i \leq m$  since by assumption, the environment is connected. Regarding  $D_m$  as a function of the variables  $\epsilon_i := \sum_j S_n(i, j) = \sum_j [P_n(i, j) + Q_n(i, j)]$ , i.e.  $D_m = D_m(\epsilon_1, \dots, \epsilon_m)$ , it is clear that  $D_m(0, \dots, 0) = 0$  and  $\frac{\partial D_m}{\partial \epsilon_i} = D_{m-1}^{(i)}$  where  $D_{m-1}^{(i)}$  is the determinant of the matrix  $S_n^{(i)}$  obtained from  $S_n$  by deleting the  $i$ th row and column. Now, the relation  $D_m > 0$  can be shown by induction. Obviously,  $D_1 > 1$ . Assuming that  $D_{m-1}^{(i)} = \frac{\partial D_m}{\partial \epsilon_i} > 0$  for  $1 \leq i \leq m$  and taking into account that connectedness implies  $\sum_i \epsilon_i > 0$ , it follows that  $D_m > 0$ . Thus  $\det S_n$ , as well as the diagonal elements of  $S_n^{-1}$  are positive. Using this result, the relations  $S_n^{-1}(i, j) \geq 0$  for  $i \neq j$  can be shown again by induction in a straightforward way.

Relation (7) and equation (5) imply that  $\Delta S_{k-1} \equiv \tilde{S}_{k-1} - S_{k-1} = -P_{k-1} S_k^{-1} Q_k \leq 0$ . In components,

$$\Delta R_{k-1}(i, j) \geq 0 \quad (i \neq j), \quad \Delta S_{k-1}(i, i) \leq 0. \quad (8)$$

From these relations we obtain  $\sum_j \Delta P_{k-1}(i, j) \leq 0$ , where we have used  $\Delta Q_{k-1} = 0$ . Similarly, we obtain:  $\Delta R_{k+1}(i, j) \geq 0, i \neq j$  and  $\sum_j \Delta Q_{k+1}(i, j) \leq 0$ . Thus, the intra-layer transition rates are non-decreasing, while the sum of rates of inter-layer jumps starting from a given site is non-increasing under a renormalization step.

Let us introduce the quantity  $\Omega_n := 1/\|S_n^{-1}\|$ , where the matrix norm  $\|\cdot\|$  is defined as  $\|A\| := \max_i \sum_j |A(i, j)|$ . From equation (5), we have  $\tilde{S}_{k-1}^{-1} = S_{k-1}^{-1} + S_{k-1}^{-1} P_{k-1} S_k^{-1} Q_k \tilde{S}_{k-1}^{-1}$ . As relation (7) is valid also for the renormalized matrices, i.e.  $\tilde{S}_{k-1}^{-1}, \tilde{S}_{k+1}^{-1} \geq 0$ , both terms on the right-hand side are non-negative, therefore  $\|\tilde{S}_{k-1}^{-1}\| = \|S_{k-1}^{-1} + S_{k-1}^{-1} P_{k-1} S_k^{-1} Q_k \tilde{S}_{k-1}^{-1}\| \geq \|S_{k-1}^{-1}\|$ , or, equivalently,  $\tilde{\Omega}_{k-1} \leq \Omega_{k-1}$ . By a similar calculation we obtain that  $\tilde{\Omega}_{k+1} \leq \Omega_{k+1}$ . The RG procedure for finite  $L$  is defined as follows. The layer with the actually largest  $\Omega_n$  is decimated, which results in a RWRE on a one layer shorter strip with effective rates given by equations (3)–(6) and the remaining  $\pi_n$  unchanged. This step is then iterated until a single layer is left. The variable defined by  $\Omega := \max_n \Omega_n$ , where  $n$  runs through the set of indices of

non-decimated (or active) layers, decreases monotonously in the course of the procedure. For the special case  $m = 1$  (1D),  $\Omega_k = Q_k(1, 1) + P_k(1, 1)$  and the transformation rules reduce to

$$\tilde{P}_{k-1}(1, 1) = \frac{P_{k-1}(1, 1)P_k(1, 1)}{Q_k(1, 1) + P_k(1, 1)}, \quad \tilde{Q}_{k+1}(1, 1) = \frac{Q_{k+1}(1, 1)Q_k(1, 1)}{Q_k(1, 1) + P_k(1, 1)}, \quad (9)$$

which have already been obtained in the context of the zero-range process [13].

The procedure described so far applies to any connected environment; as a trivial case even to the homogeneous environment. From now on we assume that the triples  $(P_n, Q_n, R_n)$  are independent, identically distributed random variables. We consider an infinite sequence of triples  $\{(P_n, Q_n, R_n)\}$  and, in the usual continuum formulation [15] of the above RG procedure, we are interested in the asymptotic scaling of  $\Omega$  with the length scale  $\xi_\Omega$  that is given by the inverse of the number density  $c_\Omega$  of active layers:  $\xi_\Omega \equiv 1/c_\Omega$ .

### 3.1. Recurrent case

First, we focus on the case of transition rate distributions for which the random walk is recurrent in almost every environment. The question of recurrence is in general non-trivial for  $m > 1$  [8, 12]; nevertheless, a sufficient condition of recurrence is that the distribution of jump rates is invariant under the interchange of  $P_n$  and  $Q_n$  [14]. Furthermore, we do not deal with special environments which lead to normal diffusive behaviour (e.g. the case  $P_n = Q_n$  for all  $n$ ). Instead, we consider less restricted situations: for instance, distributions where  $P_n$  and  $Q_n$  are independent. In this case, the above special environments form only a zero-measure set in the limit  $L \rightarrow \infty$ .

As a first step, we investigate the limits of transition rates when the density of active layers  $c_\Omega$  goes to zero. Consider a site  $(n, i)$  in an active layer in an arbitrary stadium of the RG procedure and assume that the initial matrix elements  $S_n(i, j)$  were renormalized to some  $\tilde{S}_n(i, j) \leq S_n(i, j)$ . Then we can write  $\sum_{j \neq i} \tilde{R}_n(i, j) \leq \sum_{j \neq i} \tilde{R}_n(i, j) + \sum_j [\tilde{P}_n(i, j) + \tilde{Q}_n(i, j)] \equiv \tilde{S}_n(i, i) \leq S_n(i, i)$ . Consequently, the intra-layer rates remain bounded throughout the RG procedure. Writing, e.g., equation (5) in the form  $\Delta S_{k-1} = -P_{k-1}S_k^{-1}Q_k$ , we see that at least one of the sets of matrices  $\{P_n\}$  and  $\{Q_n\}$  must tend to zero as  $c_\Omega \rightarrow 0$ , otherwise the matrices  $S_n$  would not remain bounded. Furthermore, it is clear that the assumption on recurrence requires that both  $\{P_n\}$  and  $\{Q_n\}$  must tend to zero if  $c_\Omega \rightarrow 0$ . This also implies that, in that limit,  $\det S_n \rightarrow 0$  and  $\Omega \rightarrow \Omega^* = 0$ . So, as the RG transformation progresses the inter-layer rates at the non-decimated layers are approaching zero without limits.

For the study of various quantities close to the fixed point  $\Omega^* = 0$ , it is expedient to define the following relation:  $f \simeq g$  if  $\lim_{\Omega \rightarrow 0} f/g = 1$ . According to the above, we have  $\tilde{S}_{k-1} \simeq S_{k-1}$  and similarly, for the matrix  $S_n^{-1} := S_n^{-1}/\|S_n^{-1}\|$ ,  $\tilde{S}_{k-1}^{-1} \simeq S_{k-1}^{-1}$  holds. One can easily show that the rows of  $\tilde{S}_n^{-1}$  are asymptotically identical, i.e.  $\tilde{S}_n^{-1}(i, j) \simeq \tilde{S}_n^{-1}(k, j)$  for  $1 \leq i, j, k \leq m$ , and the vectors formed from the rows tend to the stationary measure  $\tilde{\pi}_n$  of the isolated  $n$ th layer, i.e.  $\tilde{S}_n^{-1}(i, j) \simeq \tilde{\pi}_n(j)$  for  $1 \leq i, j \leq m$ , where  $\tilde{\pi}_n$  is the solution of the equation  $\tilde{\pi}_n \tilde{R}_n = 0$  which fulfils the condition  $\sum_i \tilde{\pi}_n(i) = 1$ . Although, the layers were not assumed to be connected within themselves initially, after many decimations they become almost surely connected due to the generated positive intra-layer transition rates when eliminating adjacent layers. If it is the case, the measure  $\tilde{\pi}_n$  is unique. Introducing the matrices  $\mathcal{P}_n := S_n^{-1}P_n$  and  $\mathcal{Q}_n := S_n^{-1}Q_n$ , equation (3) can be written as  $\tilde{\mathcal{P}}_{k-1} - \tilde{\mathcal{P}}_{k-1}\Delta_{k-1} = \mathcal{P}_{k-1}\mathcal{P}_k/\Omega_k$  with  $\Delta_k \equiv \tilde{S}_{k-1}^{-1} - S_{k-1}^{-1}$ . Using equation (1) we obtain that  $\|S_k^{-1}(P_k + Q_k)\| = 1$ . The rows of  $S_k^{-1}$  are asymptotically identical, therefore  $\|S_k^{-1}P_k\| + \|S_k^{-1}Q_k\| \simeq \|S_k^{-1}(P_k + Q_k)\| = 1$  and  $\Omega_k \simeq \|P_k\| + \|Q_k\|$ . Furthermore,  $\Delta_k \rightarrow 0$  if

$\Omega \rightarrow 0$ , thus we obtain the asymptotical renormalization rule  $\tilde{\mathcal{P}}_{k-1} \simeq \mathcal{P}_{k-1} \mathcal{P}_k / (\|\mathcal{P}_k\| + \|\mathcal{Q}_k\|)$ , and we have a similar equation for  $\tilde{\mathcal{Q}}_{k+1}$ . Using that the rows of  $\|\mathcal{P}_k\|$  are asymptotically identical, we have  $\|\mathcal{P}_{k-1} \mathcal{P}_k\| \simeq \|\mathcal{P}_{k-1}\| \cdot \|\mathcal{P}_k\|$  and obtain finally

$$\|\tilde{\mathcal{P}}_{k-1}\| \simeq \frac{\|\mathcal{P}_{k-1}\| \cdot \|\mathcal{P}_k\|}{\|\mathcal{P}_k\| + \|\mathcal{Q}_k\|}, \quad \|\tilde{\mathcal{Q}}_{k+1}\| \simeq \frac{\|\mathcal{Q}_{k+1}\| \cdot \|\mathcal{Q}_k\|}{\|\mathcal{P}_k\| + \|\mathcal{Q}_k\|}. \quad (10)$$

We see that these equations have the same form as those of the 1D RWRE in equation (9). The physical interpretation of these results is clear. If  $\Omega \ll 1$ , the effective inter-layer rates are much smaller than the effective intra-layer rates, thus the walker in the renormalized environment spends very long time in a layer until it jumps to another one, so that its quasistationary distribution within the layer is given asymptotically by  $\tilde{\pi}_n$ . When the walker leaves the layer it does not ‘remember’ at which site it entered the layer and irrespectively of this site, the effective jump rates to the adjacent layer to the right and left are  $\|\tilde{\mathcal{P}}_n\|$  and  $\|\tilde{\mathcal{Q}}_n\|$ , respectively. Thus we may say that the model under study renormalizes asymptotically to a 1D RWRE. In the course of the RG transformation, the normalization of the measure is obviously not conserved, i.e.  $\sum'_{(n,i)} \pi_n(i) < 1$ , where the prime denotes that the summation goes over the active sites. Nevertheless, on a finite strip, the walker spends most of the time in a small number of layers and the sum of  $\pi_n(i)$  over almost all sites goes to zero in the limit  $L \rightarrow \infty$ , which is closely related to the Golosov localization [5]. At any stage of the RG transformation, the layer with the maximal  $\Omega_n$  is decimated and  $\Omega_n \sum_i \pi_n(i)$  can be interpreted, at least close to the fixed point, as the probability current from the  $n$ th layer to the neighbouring ones. This ensures that layers with smaller  $\sum_i \pi_n(i)$ , i.e. where the walker can be found with a smaller probability, are decimated typically earlier in the course of the SDRG procedure. Thus, fixing the length scale  $\xi > 1$  and renormalizing a finite strip of length  $L > \xi$  to a strip of length  $L' = L/\xi$ , we expect that  $\sum'_{(n,i)} \pi_n(i) \rightarrow \mathcal{O}(1)$  almost always if  $L \rightarrow \infty$ . Now, if the correct normalization of  $\pi_n(i)$  in the renormalized strip is restored by dividing by  $\sum'_{(n,i)} \pi_n(i)$ , the probability current along the strip is modified only by an  $\mathcal{O}(1)$  factor. On the other hand, the current is invariant under the RG transformation, thus assuming that  $\xi \gg 1$ , the RWRE on a strip of length  $L$  has the same current up to an  $\mathcal{O}(1)$  factor as an effective 1D RWRE of length  $L' \sim L$ . This implies that the current of the RWRE on a strip must asymptotically scale with the size as that of the 1D RWRE. Consequently, the inverse of the current, which gives the mean time  $\tau$  that the walker needs to make a complete tour on the strip, must scale with  $L$  asymptotically just as in one dimension,

$$(\ln \tau)^2 \sim L. \quad (11)$$

Next, we have a closer look on the RG equations (10) and determine the scaling relation between  $\Omega$  and  $\xi_\Omega$  by pointing out the asymptotic equivalence to an already solved problem. In order to do this, we assume that the distributions of effective rates  $\|\mathcal{P}\|$  and  $\|\mathcal{Q}\|$  broaden on logarithmic scale without limits as  $\Omega \rightarrow 0$ . This property, which can be justified *a posteriori*, is characteristic of the so-called infinite randomness fixed points and ensures the asymptotical exactness of the procedure [15]. As a consequence, at the layer to be decimated, almost surely either  $\|\mathcal{P}_k\|/\|\mathcal{Q}_k\|$  or  $\|\mathcal{Q}_k\|/\|\mathcal{P}_k\|$  tends to zero if  $\Omega \rightarrow 0$ . In the first case,  $\Omega \simeq \|\mathcal{P}_k\| + \|\mathcal{Q}_k\| \simeq \|\mathcal{Q}_k\|$  and the decimation rules read

$$\|\tilde{\mathcal{P}}_{k-1}\| \simeq \frac{\|\mathcal{P}_{k-1}\| \cdot \|\mathcal{P}_k\|}{\|\mathcal{Q}_k\|}, \quad \|\tilde{\mathcal{Q}}_{k+1}\| \simeq \|\mathcal{Q}_{k+1}\|, \quad (12)$$

while in the second case  $\Omega \simeq \|\mathcal{P}_k\|$  and

$$\|\tilde{\mathcal{P}}_{k-1}\| \simeq \|\mathcal{P}_{k-1}\|, \quad \|\tilde{\mathcal{Q}}_{k+1}\| \simeq \frac{\|\mathcal{Q}_{k+1}\| \cdot \|\mathcal{P}_k\|}{\|\mathcal{P}_k\|}. \quad (13)$$

For the above transformation rules, it has been shown in [15] in the continuum limit that the distributions of  $\|\mathcal{P}\|$  and  $\|\mathcal{Q}\|$  flow in the recurrent case (apart from some singular initial distributions) to the strongly attractive self-dual fixed point with identical distribution of  $\|\mathcal{P}\|$  and  $\|\mathcal{Q}\|$ :  $\rho^*(\eta) = e^{-\eta}\Theta(\eta)$ , where  $\eta \equiv \ln(\Omega/\|\mathcal{P}\|)/\ln(\Omega_0/\Omega)$ ,  $\Omega_0$  is the initial value of  $\Omega$  and  $\Theta(x)$  is the Heaviside step function. Furthermore, the asymptotic scaling relation between  $\xi_\Omega$  and  $\Omega$  reads

$$\xi_\Omega \sim \ln^2(\Omega_0/\Omega). \quad (14)$$

Carrying out the RG transformation in a finite but long strip up to the last layer which is indexed by  $l$ , the magnitude of the current can be written as  $|J| = |\pi_l(\hat{P}_l - \hat{Q}_l)| \approx \sum_i \pi_l(i)|(\|\mathcal{P}_l\| - \|\mathcal{Q}_l\|)| \sim \sum_i \pi_l(i)\Omega_l$ , where we used in the last step that for large  $L$ ,  $\|\mathcal{P}_l\|$  and  $\|\mathcal{Q}_l\|$  differ typically by many orders of magnitude. Taking into account that  $\sum_i \pi_l(i)$  is expected to remain finite for almost all environments in the limit  $L \rightarrow \infty$  and substituting  $L$  for the length scale in equation (14) we arrive again at equation (11). From this scaling relation we conclude that the typical displacement of the first coordinate  $x$  of the walker on an infinite strip scales with the time in the recurrent case as  $x \sim (\ln t)^2$  for almost all environments.

### 3.2. Sub-linear transient phase

Now, we consider the case when the environment is still an independent, identically distributed sequence but the random walk is transient. It is known for the 1D RWRE that if  $0 < \mu_1 < 1$ , where  $\mu_1$  is the unique positive root of the equation  $[\overline{Q(1, 1)/P(1, 1)}]^{\mu_1} = 1$  and the overbar denotes averaging over the distributions of  $Q(1, 1)$  and  $P(1, 1)$ , the displacement grows sub-linearly as  $x \sim t^{\mu_1}$  [1, 16]. In the analogous zero-velocity transient phase of the RWRE on a strip, the matrices  $P_n$  and  $Q_n$  must still renormalize to zero, and the asymptotical transformation rules are given by equations (12)–(13). The analysis of these RG equations in the continuum limit has been carried out in [17] and has yielded the asymptotical result:  $\xi_\Omega \sim (\Omega/\Omega_0)^{-\mu}$ . We thus conclude that the displacement grows as  $x \sim t^\mu$  also for the RWRE on a strip in this phase. For the 1D RWRE,  $\mu = \mu_1$ , which is due to the fact that the energy landscape defined by  $U_{n+1} - U_n = \ln[Q_{n+1}(1, 1)/P_n(1, 1)]$  carries the full information on  $\mu_1$  and even the approximative rules in equations (12)–(13) leave the energy difference between active sites invariant (cf the method in [6]). For  $m > 1$ , equations (12)–(13) are valid only asymptotically and the problem how the exponent  $\mu$  is related to the initial distribution of jump rates is out of the scope of this approach.

## 4. Discussion

We have presented in this work an SDRG scheme for the RWRE on quasi-one-dimensional lattices, which incorporates also the RWRE with bounded non-nearest neighbour jumps. We have made use of that by eliminating appropriately chosen groups of lattice sites, the topology of the network of transitions remains invariant. We mention that there are special sub-networks of transitions with positive rates which are invariant under the transformation: As can be seen from equations (3)–(4), if the  $i$ th row or column of  $P_n$  or  $Q_n$  is zero for all  $n$ , then this remains valid also after an RG step. An example for  $m = 2$  is the process with the only positive inter-layer rates  $P_n(1, 1)$  and  $Q_n(2, 2)$ , which can be interpreted as a 1D persistent RWRE. We have shown that the model renormalizes to an effective 1D RWRE and concluded that, although, the finite-size corrections are strong (see [10]), Sinai scaling is valid asymptotically in the recurrent case, while in the sub-linear transient regime the displacement grows as  $x \sim t^\mu$ . Although, the method is not appropriate for establishing an analytical relation between the non-universal

exponent  $\mu$  and the initial distribution of jump rates, the numerical implementation of the exact RG scheme provides a more efficient tool for the estimation of  $\mu$  than the direct solution of equations (2).

When this work was finalized, a preprint by Bolthausen and Goldsheid appeared, in which similar results are obtained in the recurrent case in a different way [18].

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